The curse of non-rigorousness: how accumulated non-rigorousness can lead to entire nonsense

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Agenda

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Title and Scope

Definitions

Motivation

Related Work

Solution
  - Basic Idea
  - Framework
  - Applications

The Curse of Non-rigorousness

- The title of this talk is inspired by “Curse of Dimensionality” coined by *Bellman* in 1961.
- Non-rigorous methods attempt to compute an approximate solution of a given problem.
- These methods are reliable for most applications, but in some cases they compute very inaccurate results.
- When non-rigorousness is accumulated, it can be powerful enough to “put its curse on computations” and result in entire nonsense.
- In this talk, I will show how this “curse” affects computing dynamical systems and I will introduce a rigorous alternative.

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Dynamical System

- The notion of a dynamical system includes:
  1. a set of its possible states, and
  2. a rule that governs the evolution of the state in time.
- Only discrete-time dynamical systems are considered.
  - $x_{n+1} = f(x_n), f : X \to X$
- Usually, computing dynamical systems aims to:
  - find *invariant sets* and identify their properties, and/or
  - perform *bifurcation analysis*

Invariant Set

- An orbit (or a trajectory) of the map f at x_0 is
  - an ordered subset of the phase space X,
  - $Or(x_0) = \{x \in X : x = f^n(x_0), \forall n \in \mathbb{Z}\}$.
- An invariant set S is
  - $S \subseteq X : Or(x) \subseteq S, \forall x \in S$, for example
  - $x = f(x)$.
- A stable invariant set (or an attractor) is
  - an invariant set that attracts nearby orbits.

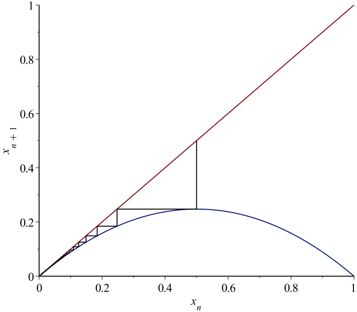
Bifurcation Analysis

- Bifurcation analysis
  - aims to find points in the parameter space at which a *qualitative change* in the system's behavior occurs,
  - is a computation extensive process, and
  - requires efficiency and adaptiveness.

Example - Invariant Set

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = rx_n(1-x_n), x \in [0,1], r \in [0,4].$

$r \in [0,1)$

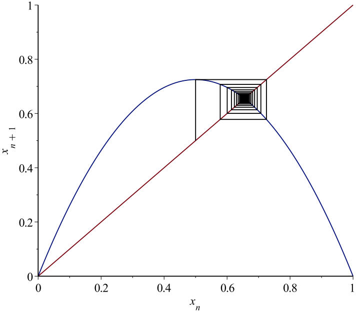


Web diagram of $\mathcal{L}_{00.99}$ showing distinction

Example - Invariant Set

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = rx_n(1 - x_n), x \in [0, 1], r \in [0, 4]$.

$r \in [1, 3)$

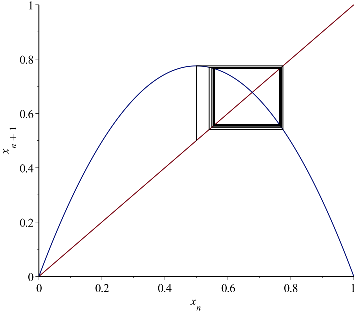


Web diagram of $\mathcal{L}_{02.90}$ showing single fixed point

Example - Invariant Set

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = rx_n(1 - x_n), x \in [0, 1], r \in [0, 4].$

$r \in [3, 1 + \sqrt{6})$

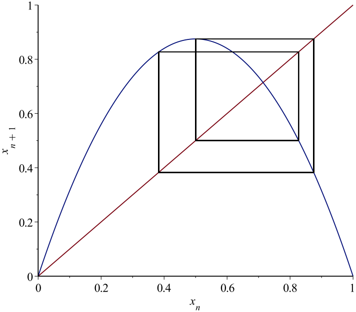


Web diagram of $\mathcal{L}_{03.10}$ showing orbit of period 2

Example - Invariant Set

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = rx_n(1 - x_n), x \in [0, 1], r \in [0, 4].$

$r \in [3.44949, 3.54409)$

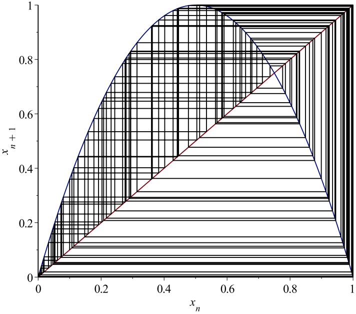


Web diagram of $\mathcal{L}_{03.50}$ showing orbit of period 4

Example - Invariant Set

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = rx_n(1 - x_n), x \in [0,1], r \in [0,4]$.

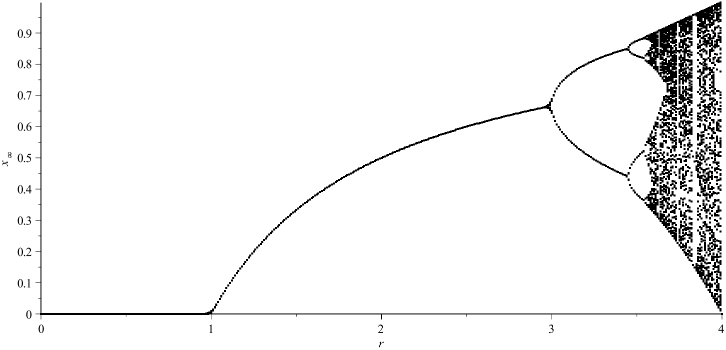
$$r > 3.57$$



Web diagram of $\mathcal{L}_{04.00}$ showing chaos

Example - Bifurcation Analysis

- A typical example is the logistic map
  - $x_{n+1} = \mathcal{L}_r(x_n) = r x_n (1 - x_n), x \in [0, 1], r \in [0, 4]$.



Bifurcation diagram of the logistic map

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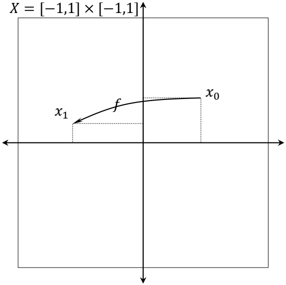
Non-rigorous Simulation

- Consider a hypothetical dynamical system
  - $x_{n+1} = f(x_n), f : X \to X, X = [-1, 1] \times [-1, 1]$

State Representation

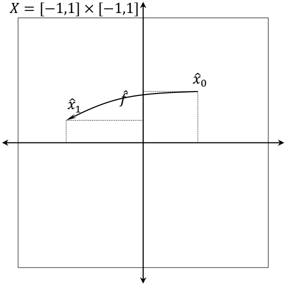


Continuous Phase Space

Non-rigorous Simulation

- Consider a hypothetical dynamical system
  - $x_{n+1} = f(x_n), f : X \to X, X = [-1, 1] \times [-1, 1]$

Map Evaluation

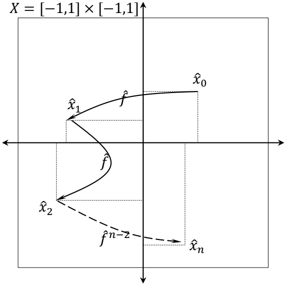


Continuous Phase Space

Non-rigorous Simulation

- Consider a hypothetical dynamical system
  - $x_{n+1} = f(x_n), f : X \to X, X = [-1, 1] \times [-1, 1]$

Time



Continuous Phase Space

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Related Work

- In 1960, *Ulam* was the first to introduce the idea of using finite resolution in computing dynamical systems.
- Since that time, a lot of work was done for computing dynamical systems at finite resolution.
  - The use of interval arithmetic and floating-point numbers is a common factor among existing work.
  - Constructing a non-ideal representation is another common factor among existing work.
- There are two major directions for the finite representation of dynamical systems using either
  - *partition* or
  - *open cover*.

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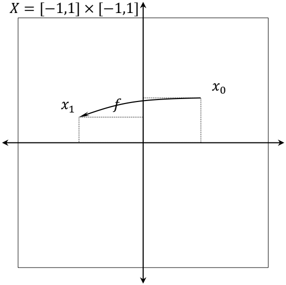
Discretization

- Discretize the phase space
  - $\mathscr{P} = \{P_i, i \in 1 \cdots n\}$

Continuous Space
Continuous Map

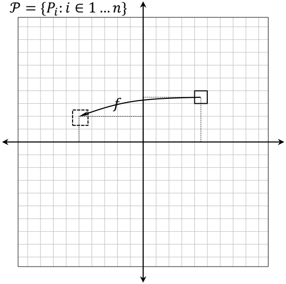


Continuous Phase Space

Discretization

- Discretize the phase space
  - $\mathscr{P} = \{P_i, i \in 1 \cdots n\}$

Discrete Space
Continuous Map

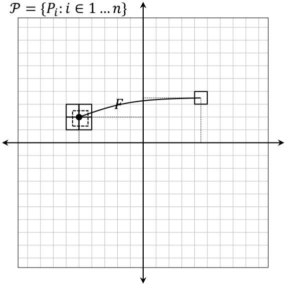


Discretized Phase Space

Discretization

- Discretize the phase space
  - $\mathscr{P} = \{P_i, i \in 1 \cdots n\}$

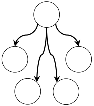
Discrete Space
Combinatorial Map



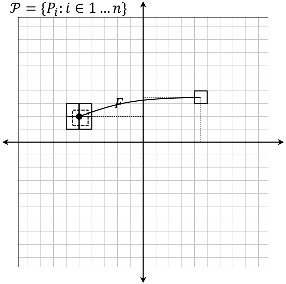
Discretized Phase Space

Advantages of Discretization

- Finite resolution is better because
  - it is computer friendly,



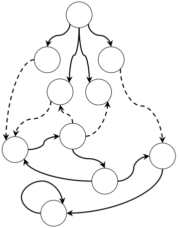
Combinatorial Representation of The Map



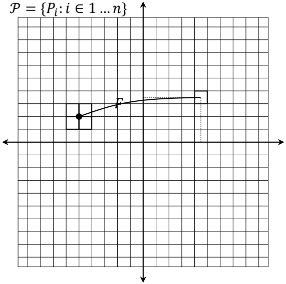
Discretized Phase Space

Advantages of Discretization

- Finite resolution is better because
    - the whole phase space can be observed, and



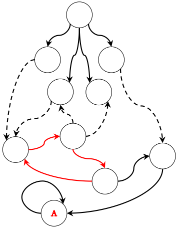
Combinatorial Representation of The Map



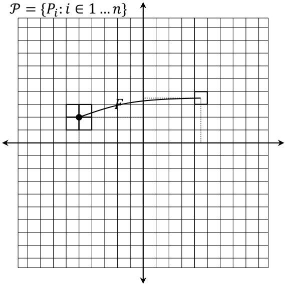
Discretized Phase Space

Advantages of Discretization

- Finite resolution is better because
  - it supports better analysis.



Combinatorial Representation of The Map



Discretized Phase Space

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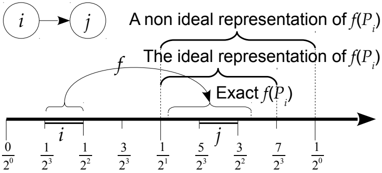
Improved Framework

- The three major changes in the improved framework are
  - rational numbers vs. floating-point numbers,
  - partition with entirely disjoint elements vs. open cover, and
  - iterative strategy vs. static strategy.
- This resulted in a more rigorous framework that features
  - independence from initial conditions,
  - ideal representation,
  - transparency, and
  - efficiency.



An example of discretized phase space for 1D map showing ideal vs. non ideal representations using dyadic partition

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Framework Applications

- Interesting Applications
  - Bifurcations analysis
  - Combinatorial Lyapunov exponent
  - Combinatorial RNG

Bifurcation Analysis

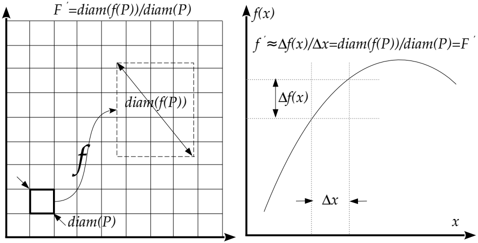
- Assuming monotonicity, an extended version of binary search was successfully used to identify bifurcation points.
- The interval of interest was iteratively divided and the map was examined until a bifurcation point is identified with the desired precision.

Quantifying Chaos

- The term “chaos” is used to describe deterministic behavior that seems random.
- Chaos quantifiers evaluate the chaoticness to a real number.
- Lyapunov exponent:
  - $\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log|f'(x_i)|$
- Combinatorial Lyapunov exponent
  - $\Lambda = \sum_{P\in\mathscr{P}} W_P \log|\mathcal{F}'(P)|$

Quantifying Chaos (cont.)

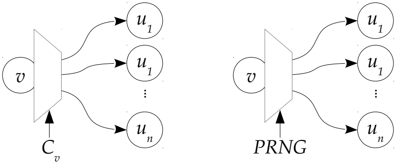
- Combinatorial derivative
  - $\mathcal{F}'(P) = \frac{\text{diam}(f(P))}{\text{diam}(P)}$
  - $f'(x) = \mathcal{F}'(P) + \mathcal{O}(\text{diam}(P)), x \in P$



Combinatorial Derivative

Combinatorial RNG

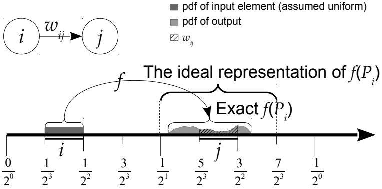
- Based on “random walk” on the combinatorial representation
- CPRNG with memory
  - Transition counters
- Memoryless CPRNG
  - Coupled with a PRNG



Combinatorial Random Number Generators

Questions

Thanks!



An example of discretized phase space for perturbed 1D map showing the transition probability